

On Hilbert's 13th Problem*

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Abstract

Every continuous function of two or more real variables can be written as the superposition of continuous functions of one real variable along with addition.

1 Introduction

The 13th Problem from Hilbert's famous list [3] asks whether every continuous function of three variables can be written as a superposition (in other words, composition) of continuous functions of two variables. Hilbert anticipated a negative answer saying,

“it is probable that the root of the equation of the seventh degree is a function of its coefficients which [...] cannot be constructed by a finite number of insertions of functions of two arguments. In order to prove this, the proof would be necessary that the equation of the seventh degree $f^7 + xf^3 + yf^2 + zf + 1 = 0$ is not solvable with the help of any continuous functions of only two arguments.”

It took over 50 years for significant progress to be made on Hilbert's 13th Problem. Then in 1954 Vitushkin [6] found a result in the direction Hilbert expected: if $n/q > n'/q'$ then there are functions of n variables with all q th order derivatives continuous which can not be written as a superposition of functions of n' variables and all q' th order derivatives continuous. In particular, there are continuously differentiable functions of three variables which can not be written as a superposition of continuously differentiable functions of two variables.

However Kolmogorov and Arnold subsequently proved a series of results culminating with Kolmogorov's 1957 Superposition Theorem.

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Theorem 1 (Kolmogorov Superposition, [4]) *For a fixed $n \geq 2$, there are $n(2n+1)$ maps $\psi^{pq} \in C([0, 1])$ such that every map $f \in C([0, 1]^n)$ can be written:*

$$f(\mathbf{x}) = \sum_{q=1}^{2n+1} g_q(\phi^q(\mathbf{x})) \quad \text{where } \phi^q(x_1, \dots, x_n) = \sum_{p=1}^n \psi^{pq}(x_p),$$

and the $g_q \in C(\mathbb{R})$ are maps depending on f .

This remarkable theorem gives a very strong positive solution to Hilbert's 13th Problem, indeed it says that every continuous function of two or more variables can be written as a superposition of continuous functions of just one variable along with just one function of two variables, namely addition.

However the Kolmogorov Superposition Theorem is not a complete solution to Hilbert's 13th Problem. Hilbert's statement of the problem explicitly refers to functions (such as the root function of an equation of the seventh degree) of three **real**, or perhaps even more naturally, complex, variables. But the Kolmogorov Superposition Theorem only deals with functions on a compact cube — the variables are restricted to a closed and bounded interval.

There have been numerous extensions to the Kolmogorov Superposition Theorem. Most notably Ostrand [5] showed that compact, finite dimensional metrizable spaces satisfy a superposition theorem, while Fridman [1] showed that the inner functions (the ψ^{pq}) can be taken to be Lipschitz. However none solve Hilbert's 13th Problem for continuous functions of three real variables.

In this paper we complete the solution of Hilbert's 13th Problem by showing that the Kolmogorov Superposition Theorem holds for all continuous functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ (Theorem 3). Further, using earlier work of the authors, [2], we characterize the topological spaces satisfying a superposition result of the Kolmogorov type. It turns out these spaces are precisely the locally compact, finite dimensional separable metrizable spaces, or equivalently, those spaces homeomorphic to a closed subspace of Euclidean space (Theorem 4).

2 Superpositions

Write $C(X, Y)$ for all continuous maps from a space X to another space Y , and $C(X)$ for $C(X, \mathbb{R})$. Note that we always use the max norm. $\|\cdot\|_\infty$, on \mathbb{R}^m .

Abstracting from Theorem 1 we make the following definition:

Definition 2 *Let X be a topological space. A family $\Phi \subseteq C(X)$ is said to be basic for X if each $f \in C(X)$ can be written: $f = \sum_{q=1}^n (g_q \circ \phi_q)$, for some ϕ_1, \dots, ϕ_n in Φ and 'co-ordinate functions' $g_1, \dots, g_n \in C(\mathbb{R})$.*

Note that the Kolmogorov Superposition Theorem says that every cube $[0, 1]^m$ has a finite basic family in which each element of the basic family is a sum of functions of one variable.

Theorem 3 Fix m in \mathbb{N} . There exist $\psi^{pq} \in C(\mathbb{R})$, for $q = 1, 2, \dots, 2m+1$ and $p = 1, 2, \dots, m$, such that for any function $f \in C(\mathbb{R}^m)$, there can be found functions g_1, \dots, g_{2m+1} in $C(\mathbb{R})$ such that:

$$f(\mathbf{x}) = \sum_{q=1}^{2m+1} g_q(\phi^q(\mathbf{x})), \quad \text{where } \phi^q(x_1, \dots, x_m) = \psi^{1q}(x_1) + \dots + \psi^{mq}(x_m).$$

Proof. We break the proof into four parts. In the first step we define a family of ‘grids’, and approximations to the functions ψ^{pq} . Next we define the ψ^{pq} and ϕ^q , and establish certain useful properties of the grids and functions. In the final two steps we show that the functions ϕ^q are basic for \mathbb{R}^m , first for compactly supported functions, and then in general.

1. Construction of the Grids and Approximations We establish by induction on k , the existence for each $k \in \mathbb{N}$, $p = 1, 2, \dots, m$, and $q = 1, 2, \dots, 2m+1$, of positive ϵ_k , $\gamma_k < 1/10$, distinct positive prime numbers $P_k^{pq} > m+10$, discrete families (‘grids’) \mathcal{S}_k^q of open intervals of \mathbb{R} and continuous functions $f_k^{pq} : \mathbb{R} \rightarrow \mathbb{R}$ such that:

- (1) the sequences of ϵ_k ’s and γ_k ’s both strictly decrease to zero (in fact, for all k , $0 < \epsilon_{k+1} < \epsilon_k/6$ and $0 < \gamma_k < 1/k$),
- (2) each member of \mathcal{S}_k^q has diameter $\leq \gamma_k$,
for each fixed k any two of the families $\{\mathcal{S}_k^q : q = 1, \dots, 2m+1\}$ cover $[-k, k]$, and all cover $\{-k, 0, k\}$;
- (3) $m\epsilon_k < 1/\prod_{p=1}^m P_k^{pq}$ for each $q = 1, 2, \dots, 2m+1$;
- (4) f_k^{pq} is non-decreasing on \mathbb{R}^+ , non-increasing on \mathbb{R}^- and constant outside $[-k, k]$;
- (5) f_k^{pq} is constant on each member of \mathcal{S}_k^q with value a positive integral multiple of $1/P_k^{pq}$, and $(f_k^{pq}(J_1) - f_k^{pq}(J_2))P_k^{pq} \not\equiv 0 \pmod{P_k^{pq}}$ given $J_1, J_2 \in \mathcal{S}_k^{pq}$; additionally, if J is an interval containing 0, then f_k^{pq} maps J to 0;
- (6) $|f_k^{pq}(k) - k| < 1/(m+1)$ and $|f_k^{pq}(-k) - k| < 1/(m+1)$;
- (7) for each $\ell \leq j < k$ and $x \in [-\ell, \ell]$, $f_j^{pq}(x) \leq f_k^{pq}(x) \leq f_j^{pq}(x) + \epsilon_j - \epsilon_k$.

Base Step: It is straightforward to find discrete collections of open intervals \mathcal{S}_1^{pq} for $p = 1, \dots, m$ and $q = 1, \dots, 2m+1$ such that any two of the families $\{\mathcal{S}_1^{pq} : q = 1, 2, \dots, 2m+1\}$ cover $[-1, 1]$, each of the families covers $\{1, 0, -1\}$, and each interval in the collection has length $\leq \gamma_1 = 1/10$.

Let n_1 be the number of all the open interval in all the collections \mathcal{S}_1^{pq} ($1 \leq p \leq m$, $1 \leq q \leq 2m+1$). For $p = 1, \dots, m$ and $q = 1, \dots, 2m+1$ pick distinct primes P_1^{pq} larger than n_1 .

Now we define f_1^{pq} on $[-1, 1]$. Then for $x > 1$ define $f_1^{pq}(x) = f_1^{pq}(1)$, and for $x < -1$ define $f_1^{pq}(x) = f_1^{pq}(-1)$.

If $J \in \mathcal{S}_1^{pq}$, then define f_1^{pq} such that f_1^{pq} restricted to J is a positive integral multiple of $1/P_1^{pq}$. More specifically, if $0 \in J$ then $f_1^{pq}(J) = 0$; if $1 \in J$ then $f_1^{pq}(J) = 1 - 1/P_1^{pq}$; and if $-1 \in J$ then $f_1^{pq}(J) = 1 - 2/P_1^{pq}$. This can easily be done so that f_1^{pq} (as defined so far) is non-decreasing on $[0, 1]$ and non-increasing on $[-1, 0]$.

For x in $[-1, 1] \setminus \bigcup \mathcal{S}_1^{pq}$, interpolate f_1^{pq} linearly.

Choose $\epsilon_1 > 0$ such that $m\epsilon_1 < 1/\prod_{p=1}^m P_1^{pq}$ for each $q = 1, 2, \dots, 2m+1$.

All (applicable) conditions (1)–(7) hold.

Inductive Step: Suppose P_{k-1}^{pq} , ϵ_{k-1} , γ_{k-1} , \mathcal{S}_{k-1}^q and f_{k-1}^{pq} are all given and satisfy the requirements (1)–(7).

By uniform continuity of f_{k-1}^{pq} on $[-(k-1), k-1]$, there exists $\gamma_k < \min\{1/k, \gamma_{k-1}\}$ such that $|f_{k-1}^{pq}(x_1) - f_{k-1}^{pq}(x_2)| < \epsilon_{k-1}/6$ if $|x_1 - x_2| < \gamma_k$ for each $p = 1, \dots, m$ and $q = 1, \dots, 2m+1$.

Then it is straightforward to find discrete collections of open intervals, \mathcal{S}_k^{pq} for $1 \leq p \leq m$ and $1 \leq q \leq 2m+1$, such that any two of the families $\{\mathcal{S}_k^{pq} : q = 1, 2, \dots, 2m+1\}$ cover $[-k, k]$, each of the families covers $\{k, 0, -k\}$, each interval in the collection has length $\leq \gamma_k$ and the distance between each pair of adjacent intervals is also $\leq \gamma_k$.

Let n_k be the total number of open intervals in all the collections \mathcal{S}_k^{pq} for $p = 1, 2, \dots, m$ and $q = 1, 2, \dots, 2m+1$. For each p, q select distinct primes P_k^{pq} so that $2n_k/P_k^{pq} < \epsilon_{k-1}/6$.

Next, we give the construction of f_k^{pq} on $[-k, k]$. Outside of $[-k, k]$ extend constantly (as in the Base Step).

- If $J \in \mathcal{S}_k^{pq}$, then $f_k^{pq}(J)$ is a positive integral multiple of $1/P_k^{pq}$. For any $J \in \mathcal{S}_k^{pq}$ with $J \cap [-(k-1), k-1] \neq \emptyset$, we can ensure that $f_{k-1}^{pq}(x) < f_k^{pq}(x) < f_{k-1}^{pq}(x) + \epsilon_{k-1}/3$.
 [i] Since $2n_k/P_k^{pq} < \epsilon_{k-1}/6$ and $|f_{k-1}^{pq}(x_1) - f_{k-1}^{pq}(x_2)| < \epsilon_{k-1}/6$ when $|x_1 - x_2| < \gamma_k$, there are $2n_k$ possible choices for the value of $f_k^{pq}(J)$ ($J \in \mathcal{S}_k^{pq}$) which makes $f_{k-1}^{pq}(x) < f_k^{pq}(x) < f_{k-1}^{pq}(x) + \epsilon_{k-1}/3$ for $x \in J \cap [-(k-1), k-1]$. As there are many fewer than $2n_k$ elements in \mathcal{S}_k^{pq} , we can select the $f_k^{pq}(J)$'s such that $(f_k^{pq}(J_1) - f_k^{pq}(J_2))P_k^{pq} \bmod P_k^{pq} \neq 0$ for any $J_1, J_2 \in \mathcal{S}_k^{pq}$.
 [ii] More specifically, if $0 \in J$ then $f_k^{pq}(J) = 0$, if $k \in J$ then $f_k^{pq}(J) = 1 - 1/P_k^{pq}$, and if $-k \in J$ then $f_k^{pq}(J) = 1 - 2/P_k^{pq}$. This can easily be done to make f_k^{pq} (as defined so far) non-decreasing on $[0, k]$ and non-increasing on $[-k, 0]$.
- If $x \notin \bigcup \mathcal{S}_k^{pq}$, let J_L and J_R be the adjacent intervals in \mathcal{S}_k^{pq} such that x lies between them. Let x_L be the right endpoint of J_L and x_R be the left end point of J_R . Then f_k^{pq} maps $[x_L, x_R]$ linearly to

$[f_{k-1}^{pq}(J_L), f_{k-1}^{pq}(J_R)]$. Since $|x_L - x_R| < \gamma_k$, $|f_{k-1}^{pq}(x_L) - f_{k-1}^{pq}(x_R)| < \epsilon_{k-1}/6$, therefore, $f_k^{pq}(x) - f_{k-1}^{pq}(x) < \epsilon_{k-1}/3 + \epsilon_{k-1}/6 = \epsilon_{k-1}/2$.

Choose ϵ_k such that $m\epsilon_k < \min\{1/\prod_{p=1}^m P_k^{pq}, \epsilon_{k-1}/6\}$ for all $1 \leq q \leq 2m+1$.

All requirements (1)–(7) are satisfied.

2. Definition and Useful Properties of the Functions, ψ^{pq} and ϕ^q For $x \in \mathbb{R}$, let $\psi^{pq}(x) = \lim_{k \rightarrow \infty} f_k^{pq}(x)$. Now for a fixed $n \in \mathbb{N}$, and any $x \in [-n, n]$, $f_k^{pq}(x) \leq \psi^{pq}(x) \leq f_k^{pq}(x) + \epsilon_k$ for $k > n+1$. So ψ^{pq} restricted to $[-n, n]$, being the uniform limit of the f_k^{pq} for $k > n+1$, is continuous on $[-n, n]$. Therefore, ψ^{pq} is continuous on \mathbb{R} .

Also, by construction, the image of $[n, n+1]$ under ψ^{pq} is a subset of $[|n| - 1/(m+1), |n| + 1 + 1/(m+1)]$ for each $n \in \mathbb{Z}$.

Let $\phi^q(x_1, \dots, x_m) = \psi^{1q}(x_1) + \dots + \psi^{mq}(x_m)$ for $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$.

Our eventual goal is to show $\{\phi^q : q = 1, 2, \dots, 2m+1\}$ is a basic family of \mathbb{R}^m , however first, we establish some useful properties of the grids and functions.

For each q and k , let $\mathcal{J}_k^q = \{C_1 \times C_2 \times \dots \times C_m : C_p \in \mathcal{S}_k^q \text{ for each } p = 1, 2, \dots, m\}$. Then we can say the following about \mathcal{J}_k^q .

- For a fixed q and k , \mathcal{J}_k^q is a discrete collection.
- For a fixed k , any element in \mathbb{R}^m belongs to at least $m+1$ rectangles of \mathcal{J}_k^q , i.e. any $m+1$ of $\{\mathcal{J}_k^q : q = 1, \dots, 2m+1\}$ form an open cover of \mathbb{R}^m .

Let $\mathcal{U}_k^q = \{\phi^q(C) : C \in \mathcal{J}_k^q\}$. Take $C = C_1 \times C_2 \times \dots \times C_m \in \mathcal{J}_k^q$, then $\phi^q(C)$ is contained in the interval $[\sum_{p=1}^m f_k^{pq}(C_p), \sum_{p=1}^m f_k^{pq}(C_p) + m\epsilon_k]$. By condition (3) in the construction of the f_k^{pq} , these closed intervals are disjoint for each q and k . Therefore,

Claim \mathcal{U}_k^q is a discrete collection of subsets of \mathbb{R} for each q and k .

3. The ϕ^q are Basic for Compactly Supported Functions We now prove:

Claim For any compactly supported $h \in C(\mathbb{R}^m)$, there are g_1, \dots, g_{2m+1} in $C(\mathbb{R})$ such that $h = \sum_{q=1}^{2m+1} g_q \circ \phi_q$.

Fix a compactly supported $h \in C(\mathbb{R}^m)$. Choose ℓ in \mathbb{N} so that $h(\mathbf{x}) = 0$ for any \mathbf{x} outside $K = [-\ell - 1, \ell + 1]^m$.

For each integer $r \geq 0$ and $q = 1, \dots, 2m+1$, find positive k_r and continuous functions $\chi_r^q : \mathbb{R} \rightarrow \mathbb{R}$ ($k_0 = \ell$ and $\chi_1^q = 0$ for each q) such that if $h^r(\mathbf{x}) = \sum_{q=1}^{2m+1} \sum_{s=0}^r \chi_s^q(\phi^q(\mathbf{x}))$ and $M_r = \sup_{\mathbf{x} \in \mathbb{R}^m} |(h_i - h_i^r)(\mathbf{x})|$, then:

- (1) $k_{r+1} > k_r$;
- (2) if $\|\mathbf{a} - \mathbf{b}\|_\infty < m/10^{k_{r+1}}$, then $|(h - h_r)(\mathbf{a}) - (h - h_r)(\mathbf{b})| < (2m+2)^{-1} M_r$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$;

- (3) χ_{r+1}^q is constant on each member of $\mathcal{U}_{k_{r+1}}^q$;
- (4) if $C \cap (\mathbb{R}^m \setminus K) \neq \emptyset$ for $C \in \mathcal{J}_{k_{r+1}}^q$, then the value of χ_{r+1}^q on $\phi_q(C)$ is 0, otherwise, its value on $\phi_q(C)$ is $(m+1)^{-1}(h-h_r)(\mathbf{y})$ for some arbitrarily chosen element $\mathbf{y} \in C$; and
- (5) $\chi_{r+1}^q(x) \leq (m+1)^{-1}M_r$ for each $x \in \mathbb{R}$.

The k_r and χ_r^q are defined inductively on r . Also for any $\mathbf{a}, \mathbf{b} \in C \in \mathcal{J}_{k_{r+1}}^q$, $\|\mathbf{a} - \mathbf{b}\|_\infty < m/10^{k_{r+1}}$. Therefore:

- (6) for $\mathbf{x} \in \bigcup\{C : C \in \mathcal{J}_{k_{r+1}}^q\}$,

$$|(m+1)^{-1}(h-h_r)(\mathbf{x}) - \chi_{r+1}^q(\phi_q(\mathbf{x}))| < (m+1)^{-1}(2m+2)^{-1}M_r.$$

Also for each $\mathbf{x} \in \mathbb{R}^m$, there are at least $m+1$ distinct values of q such that $\mathbf{x} \in \bigcup\{C : C \in \mathcal{J}_{k_{r+1}}^q\}$. Then there are $m+1$ values of q such that (6) is true; for the other m values of q , (5) in the construction holds.

Hence, for $\mathbf{x} \in K$,

$$\begin{aligned} |(h-h_{r+1})(\mathbf{x})| &= |(h-h_r)(\mathbf{x}) - \sum_{q=1}^{2m+1} \chi_{r+1}^q(\phi_q(\mathbf{x}))| \\ &< (m+1) \cdot (m+1)^{-1}(2m+2)^{-1}M_r + m \cdot (m+1)^{-1}M_r \\ &= \frac{2m+1}{2m+2}M_r. \end{aligned}$$

While for $\mathbf{x} \notin K$, $\sum_{q=1}^{2m+1} \chi_{r+1}^q(\phi_q(\mathbf{x})) = 0$ by property (4).

Therefore, $M_{r+1} < (2m+1) \cdot (2m+2)^{-1} \cdot M_r$, so $M_r < ((2m+1) \cdot (2m+2)^{-1})^r \cdot M_0$ for each r , hence $\lim_{r \rightarrow \infty} M_r = 0$, and thus $h(\mathbf{x}) = \lim_{r \rightarrow \infty} h_r(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

Moreover, by condition (5), the functions $\sum_{s=0}^r \chi_s^q$ converge uniformly for each q to a continuous function $g_q : \mathbb{R} \rightarrow \mathbb{R}$ and

$$h(\mathbf{x}) = \lim_{r \rightarrow \infty} h_r(\mathbf{x}) = \lim_{r \rightarrow \infty} \sum_{q=1}^{2m+1} \sum_{s=0}^r \chi_s^q(\phi_q(\mathbf{x})) = \sum_{q=1}^{2m+1} g_q(\phi_q(\mathbf{x})).$$

This complete the proof of the Claim.

4. The ϕ^q are Basic for All Functions We complete the proof by showing:

Claim For any $f \in C(\mathbb{R}^m)$, there are g_1, \dots, g_{2m+1} in $C(\mathbb{R})$ such that $f = \sum_{q=1}^{2m+1} g_q \circ \phi_q$.

First some preliminary definitions. Let K_n^i be

$$\{(x_1, x_2, \dots, x_m) : x_i \in [-n-2, -n] \cup [n, n+2], x_j \in [-n-2, n+2] \text{ for } j \neq i\},$$

and let $\mathcal{K} = \{K_n = \bigcup_{i=1}^m K_n^i : n \in \mathbb{N} \cup \{0\}\}$.

For each n , the image of K_n under ϕ^q is $\{[n-1, m(n+2)+1] : n \in \mathbb{N} \cup \{0\}\}$ which is a locally finite collection of subsets of \mathbb{R} .

Next we inductively define a sequence of continuous functions α_n on \mathbb{R}^m for $n \in \mathbb{N} \cup \{0\}$, as follows:

Base step: $\alpha_0(\mathbf{x}) = 1$ for $\mathbf{x} \in [-1, 1]^m$, $\alpha_0(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^m \setminus K_0$.

Inductive step: $\alpha_n(\mathbf{x}) = 1 - \alpha_{n-1}(\mathbf{x})$ for $\mathbf{x} \in K_n \cap K_{n-1}$, $\alpha_n(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^m \setminus K_n$.

To prove the Claim, take any $f \in C(\mathbb{R}^m)$. Then $f(\mathbf{x}) = \sum_{i=0}^{\infty} \alpha_i(\mathbf{x}) \cdot f(\mathbf{x})$. Also $\alpha_i(\mathbf{x}) \cdot f(\mathbf{x}) = 0$ if $\mathbf{x} \notin K_i$.

From the Claim in the previous Step, for each $i \in \mathbb{N} \cup \{0\}$, there exist continuous functions g_1^i, \dots, g_{2m+1}^i such that $\alpha_i(\mathbf{x}) \cdot f(\mathbf{x}) = \sum_{q=1}^{2m+1} g_q^i(\phi^q(\mathbf{x}))$.

Then let $g_q = \sum_{i=0}^{\infty} g_q^i$. This function is well-defined and continuous because $\{x : g_q^i(x) \neq 0\} \subseteq [i-1, m(i+2)+1]$, which means there are only finitely many i with $g_q^i(x) \neq 0$ for each $x \in \mathbb{R}$.

Then we have

$$f(\mathbf{x}) = \sum_{i=0}^{\infty} \alpha_i(\mathbf{x}) \cdot f(\mathbf{x}) = \sum_{i=0}^{\infty} \sum_{q=1}^{2m+1} g_q^i(\phi^q(\mathbf{x})) = \sum_{q=1}^{2m+1} g_q(\phi^q(\mathbf{x})),$$

— as claimed. ■

Theorem 4 *Let X be a Tychonoff space. Then the following are equivalent:*

- (1) *some power of X has a finite basic family;*
- (2) *for every $m, n \in \mathbb{N}$, there is an $r \in \mathbb{N}$ and ψ^{pq} from $C(X, \mathbb{R}^n)$, for $q = 1, \dots, r$ and $p = 1, \dots, m$, such that every $f \in C(X^m, \mathbb{R}^n)$ can be written*

$$f(x_1, \dots, x_m) = \sum_{q=1}^r g_q \left(\sum_{p=1}^m \psi^{pq}(x_p) \right),$$

for some g_1, \dots, g_r in $C(\mathbb{R}^n, \mathbb{R}^n)$;

- (3) *X is a locally compact, finite dimensional separable metric space, or equivalently, homeomorphic to a closed subspace of Euclidean space.*

Proof. It was shown in [2] that a Tychonoff space has a finite basic family if and only if it is a locally compact, finite dimensional separable metrizable space. Hence (1) implies (3), and (2) implies (1).

Now suppose (3) holds and X is a locally compact, finite dimensional separable metric space. Fix m . Then X is (homeomorphic to) a closed subspace of some \mathbb{R}^ℓ . We establish (2) when $n = 1$. The general case follows easily by working co-ordinatewise.

According to Theorem 3 there exist ψ^{pq} for $p = 1, 2, \dots, \ell m$ and $q = 1, 2, \dots, 2\ell m + 1$ such that any $f \in C(\mathbb{R}^{\ell m})$ can be written as $f(x_1, \dots, x_{\ell m}) = \sum_{q=1}^{2\ell m+1} g_q(\sum_{p=1}^{\ell m} \psi^{pq}(x_p))$ for some $g_q \in C(\mathbb{R})$.

Let $r = 2\ell m + 1$. Let $\Psi^{pq} = \sum_{i=1+(p-1)m}^{m+(p-1)m} \psi^{iq}$ for $p = 1, \dots, m$ and $q = 1, \dots, r$. Since X is a closed subset of \mathbb{R}^ℓ , any continuous function on X can be continuously extended to \mathbb{R}^ℓ . Then $\{\Psi^{pq} \upharpoonright X : p = 1, \dots, m, \text{ and } q = 1, \dots, r\}$ are as required. ■

Note that from Theorem 4(2) it follows that every continuous function of three complex variables can be written as a superposition of addition and continuous functions of one complex variable.

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